# Harvard CS 121 and CSCI E-207

# Lecture 7: Non-Regular Languages

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- **Reading:** Sipser, §4.2 "The Diagonalization Method," pages 174–178 (from just before Definition 4.12 until just before Corollary 4.18) and §1.4.

# Countable Unions of Countable Sets

**Proposition:** The union of countably many countable sets is countable.

**Proof:**

# Are there uncountable sets?
# (Infinite but not countably infinite)

**Theorem:** $P(\mathcal{N})$ is uncountable

(The set of all sets of natural numbers)

**Proof by contradiction:**

(i.e. assume that $P(\mathcal{N})$ is countable and show that this results in a contradiction)

- Suppose that $P(\mathcal{N})$ were countable.
- Then there is an enumeration of all subsets of $\mathcal{N}$ say $P(\mathcal{N}) = \{S_0, S_1, \ldots\}$

# Diagonalization

| $S_i$ \ $j =$ | 0 | 1 | 2 | 3 | 4 | |
|---|---|---|---|---|---|---|
| $S_0$ | $Y$ | $N$ | $N$ | $Y$ | $N$ | $\cdots$ |
| $S_1$ | $N$ | $N$ | $N$ | $N$ | $N$ | $\cdots$ |
| $S_2$ | $Y$ | $Y$ | $N$ | $Y$ | $Y$ | $\cdots$ |
| $S_3$ | $N$ | $N$ | $N$ | $Y$ | $N$ | $\cdots$ |
| $\vdots$ | | | | $D$ | | |

"$Y$" in row $i$, column $j$ means $j \in S_i$

- Let $D$ = $\{i \in \mathcal{N} : i \in S_i\}$ be the diagonal.
- $D = YNNY \ldots = \{0, 3, \ldots\}$
- Let $\overline{D} = \mathcal{N} - D$ be its complement.
- $\overline{D} = NYYN \ldots = \{1, 2, \ldots\}$
- **Claim:** $\overline{D}$ is omitted from the enumeration, contradicting the assumption that every set of natural numbers is one of the $S_i$s.

  **Pf:** $\overline{D}$ is different from each row because they differ at the diagonal.

# Cardinality of Languages

- An alphabet $\Sigma$ is finite by definition
- **Proposition:** $\Sigma^*$ is countably infinite.
  **Proof:**
- So every language is either finite or countably infinite
- $P(\Sigma^*)$ is uncountable, being the set of subsets of a countable infinite set.

  i.e. There are uncountably many languages over any alphabet

  **Q:** Even if $|\Sigma| = 1$?

# Existence of Non-regular Languages

**Theorem:** For every alphabet $\Sigma$, there exists a non-regular language over $\Sigma$.

**Proof:**

- There are only countably many regular expressions over $\Sigma$.

  $\Rightarrow$ There are only countably many regular languages over $\Sigma$.

- There are uncountably many languages over $\Sigma$.
- Thus at least one language must be non-regular.

In fact, "almost all" languages must be non-regular.

# Existence of Non-regular Languages

**Theorem:** For every alphabet $\Sigma$, there exists a non-regular language over $\Sigma$.

**Q:** Could we do this proof using DFAs instead?

**Q:** Can we get our hands on an *explicit* non-regular language?

## Goal: Explicit Non-Regular Languages

It **appears** that a language such as

$$\begin{aligned} L &= \{x \in \Sigma^* : |x| = 2^n \text{ for some } n \geq 0\} \\ &= \{a, b, aa, ab, ba, bb, aaaa, \ldots, bbbb, aaaaaaaa, \ldots\} \end{aligned}$$

can't be regular because the "gaps" in the set of possible lengths become arbitrarily large, and no DFA could keep track of them.

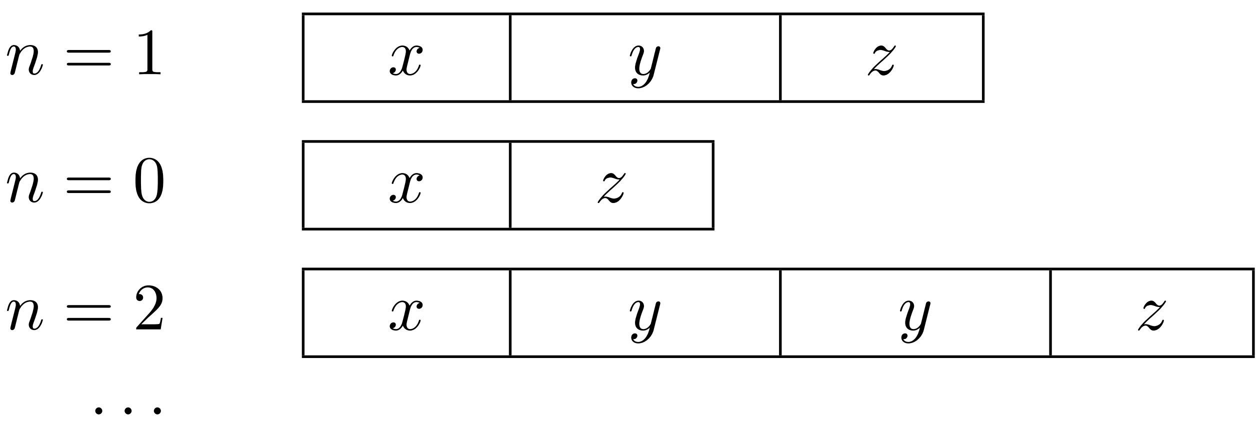
But this isn't a proof!

**Approach:**

1. Prove some general property $P$ of all regular languages.
2. Show that $L$ does not have $P$.

# Pumping Lemma (Basic Version)

If $L$ is regular, then there is a number $p$ (the pumping length) such that every string $s \in L$ of length at least $p$ can be divided into $s = xyz$, where $y \neq \varepsilon$ and for every $n \geq 0$, $xy^nz \in L$.




- Why is the part about $p$ needed?
- Why is the part about $y \neq \varepsilon$ needed?

# Pumping Lemma Example

- Consider

$$L = \{x : x \text{ has an even \# of } a\text{'s and an odd \# of } b\text{'s}\}$$

- Since $L$ is regular, pumping lemma holds.

  (i.e, every sufficiently long string $s$ in $L$ is "pumpable")

- For example, if $s = aab$, we can write $x = \varepsilon$, $y = aa$, and $z = b$.

# Pumping the even $a$'s, odd $b$'s language

- Claim: $L$ satisfies pumping lemma with pumping length $p = 4$.
- Proof:
- **Q:** Can the Pumping Lemma be used to prove that $L$ is regular?