Harvard CS 121 and CSCI E-207

Lecture 16: Decidability & The Universal TM

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October 30, 2012

- Reading: Sipser §4.1, §4.2.

A "Universal" algorithm for deciding regular languages

- **Proposition:** $A_{\text{DFA}} = \{\langle D, w \rangle : D$ a DFA that accepts $w\}$ is decidable.

  Proof sketch:

  - First check that input is of proper form.
  - Then simulate D on w. Implementation on a multitape TM:
    - Tape 2: String w with head at current position (or to be precise, its representation).
    - Tape 3: Current state q of D (i.e., its representation).

- Could work with other encodings, e.g. transition function as a matrix rather than list of triples.

Representation independence

General point: Notions of computability (e.g. decidability and recognizability) are independent of data representations.

- A TM can convert any reasonable encoding to any other reasonable encoding.
- We will use $\langle \cdot \rangle$ to mean "any reasonable encoding".
- We'll need to revisit representation issues again when we discuss computational *speed.*
- For the moment when we are interested only in whether problems are decidable, undecidable, recognizable, etc., so we can be content knowing that there is *some* representation on which an algorithm could work.

High-Level Algorithm Descriptions

Given the C–T Thesis and representation independence, we no longer need to refer to a specific computing model or or data representation when describing an algorithm. Instead:

- Describe it as a sequence of steps operating on higher-level data types (e.g. numbers, graphs, automata, grammars).
- Each step: simple enough that it is clear it can be implemented on a reasonable model (such as a TM) using a reasonable data representation.
- Freely make use of algorithms we have seen (or are well-known, such as elementary arithmetic) as subroutines.
- Freely make use of control-flow primitives, such as loops, if-then-else, gotos, etc.

More Decidable Problems

- $\{\langle R, w\rangle : R$ is a regular expression that generates $w\}$.

- $\{\langle X\rangle : X$ is an DFA/NFA/RE such that $L(X) = \emptyset\}$.

- $\{\langle X\rangle : X$ is a DFA/NFA/RE such that $|L(X)| = \infty\}$.

More Decidable Problems

- $\{\langle M, w\rangle : M$ is a PDA that accepts $w\}$.

- Any given context-free language (what does this question mean?)

A Universal Turing Machine

Theorem: There is a Turing machine U, such that when U is given $\langle M, w\rangle$ for any TM M and w, U produces the same result (accept/reject/loop) as running M on w.

Proof: Initially,

- First tape contains $\langle M\rangle$, including in particular its transition function δ_M.
- Second tape contains $\langle w\rangle$.
- Third tape contains $\langle q_{\text{start}}\rangle$.
- Simulate steps of M by multiple steps of U.

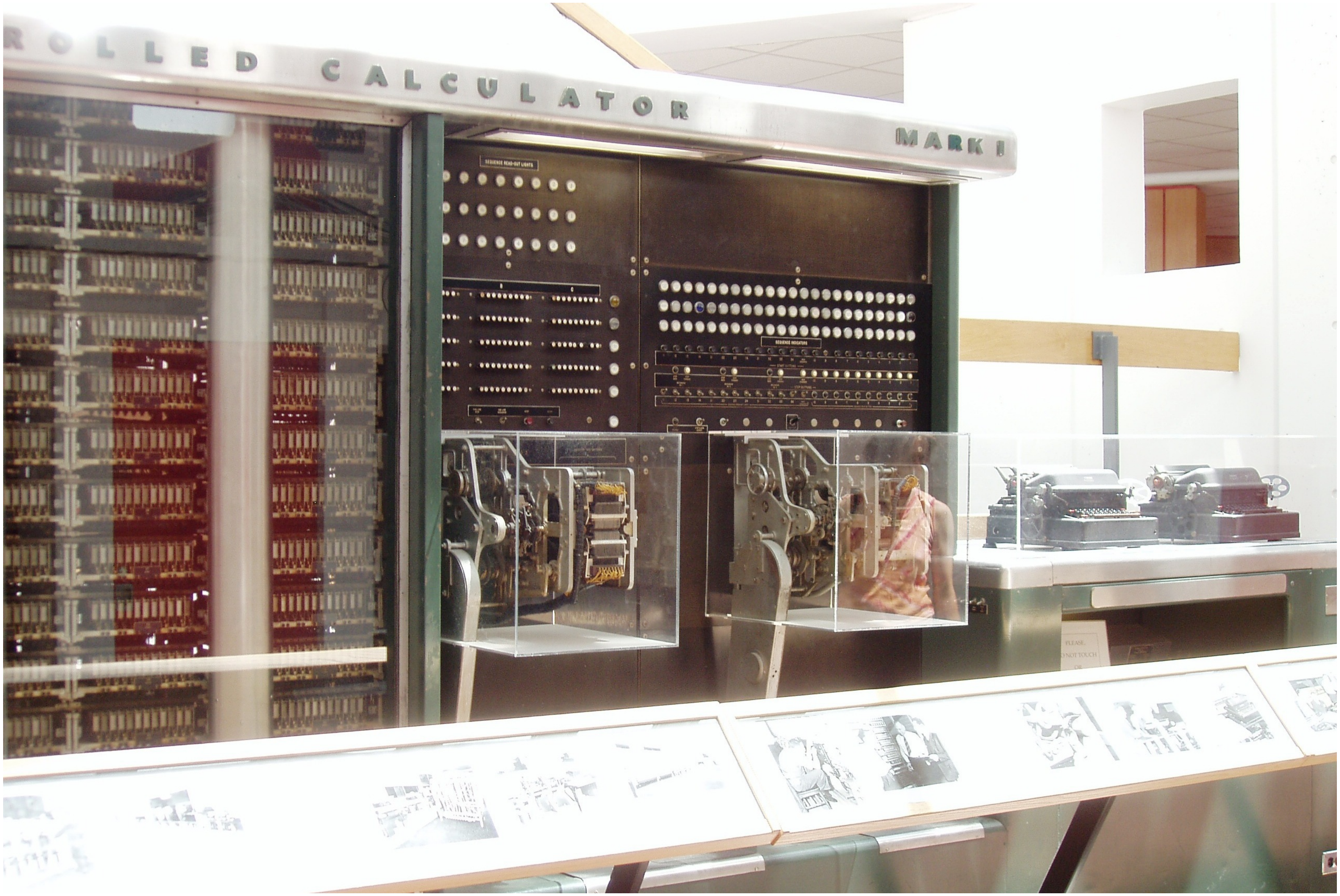
(Brief return to implementation description.)

$\Rightarrow$ Turing machines can be "programmed".

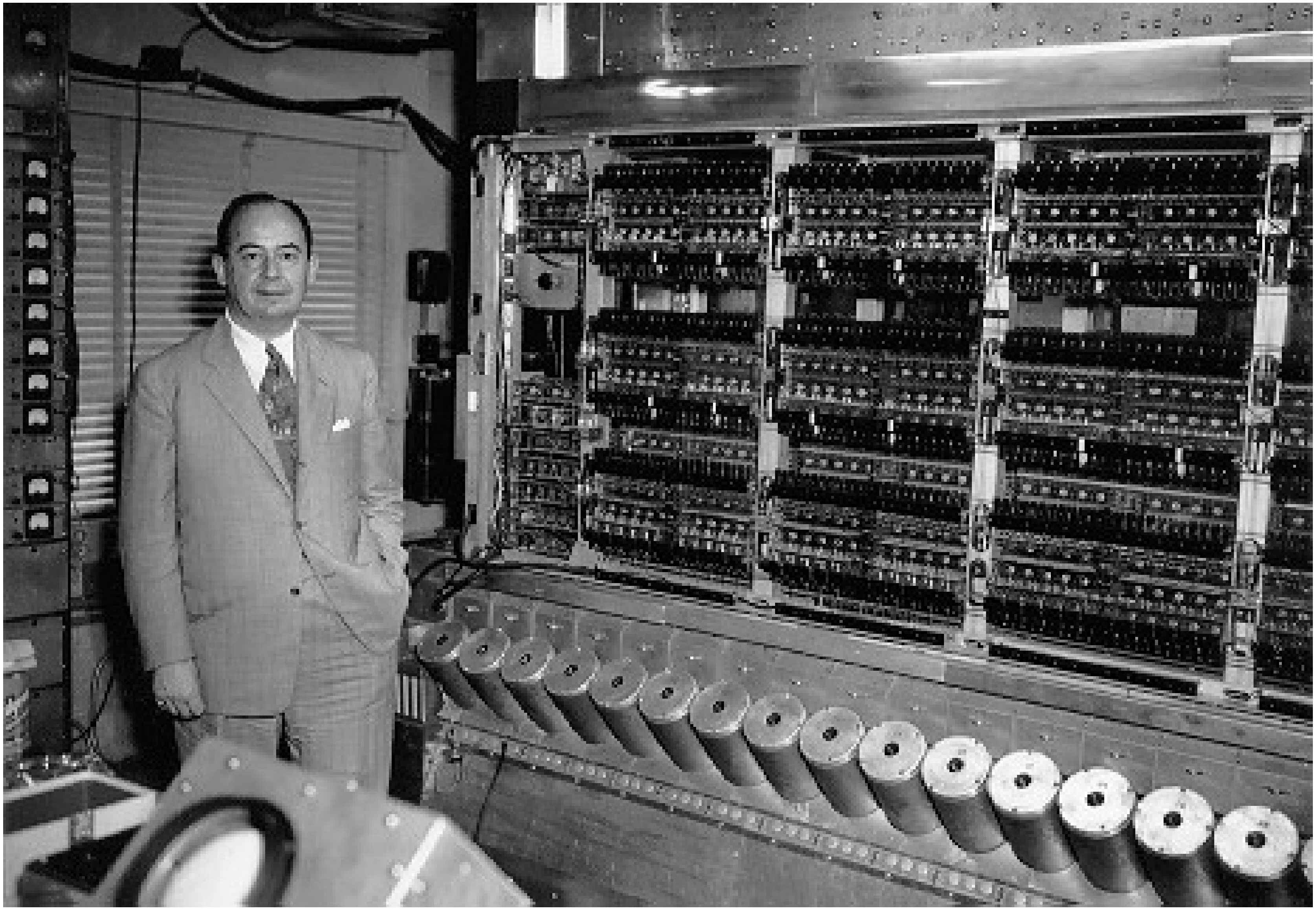
From "On Computable Numbers" (1936)

6. The universal computing machine. It is possible to invent a single machine which can be used to compute any computable sequence. If this machine I is supplied with a tape on the beginning of which is written the S.D of some computing machine M, then I will compute the same sequence as M. In this section I explain in outline the behavior of the machine. The next section is devoted to giving the complete table for I.

The Mark I (1944): "Harvard Architecture"



The Institute for Advanced Study Machine (1946-51): “Von Neumann Architecture



Technological Consequences of Universal TMs

General-purpose, programmable computers:

- Single hardware can support all computing tasks.
- Arbitrary hardware can be represented as software programs (cf. virtual machines).
- Programs can be treated like data (von Neumann architecture).

Theoretical Consequences of Universal TMs

- $\text{A}_{\text{TM}} = \{\langle M, w\rangle : M$ accepts $w\}$ is Turing-recognizable.
- $\text{HALT}_{\text{TM}} = \{\langle M, w\rangle : M$ eventually halts on $w\}$ ("The Halting Problem") is Turing-recognizable.
- **Q:** Are these sets decidable?
- **Q:** Are there undecidable languages?