CS 121, Section 0

Week of September 3, 2012

Overview

In this section, our goal is to learn/review the elements of a good proof and practice proof techniques. In doing so, we will also review some mathematical prerequisites for the course, including sets, functions, relations, strings, and graphs. Sipser Chapter 0 contains an overview of these topics as well.

Outline:

1. **Tips on Approaching Proofs.**
2. **Exercises.**

1 Tips on Approaching Proofs

A good proof consists of a sequence of *logical steps*. Start at the assumptions and proceed in small steps, each of which logically follows from the previous steps and from facts that are already known. Depending on the situation, it is generally good to give the reason or justification for each step.

1.1 First Steps

Consider the following example problem:

Exercise 1.1 (Theorem 0.20 in Sipser)**.** *Given two sets A and B that are subsets of a universe U, prove that $\overline{A \cup B} = \overline{A} \cap \overline{B}$.*

1. Carefully read the statement you want to prove. Do you understand all of the notation? Can you restate what it is you want to show?
2. Develop some intuition for the problem. Try drawing a diagram or working through a few examples. Can you see why a statement must be true or how to prove it?

3. When you believe you have found the proof, write it up properly. Does each step follow from the previous step by simple reasoning?

1.2 Writing It Up

State your game plan. A good proof begins by explaining the general line of reasoning, e.g. "We use induction" or "We argue by contradiction". This creates a rough mental picture into which the reader can fit the subsequent details.

Keep a linear flow. We sometimes see proofs that are like mathematical mosaics, with juicy tidbits of reasoning sprinkled judiciously across the page. This is not good. The steps of your argument should follow one another in a clear, sequential order.

Explain your reasoning. Many students initially write proofs the way they compute integrals. The result is a long sequence of expressions without explanation. This is bad. A good proof usually looks like an essay with some equations thrown in. Use complete sentences.

Introduce notation thoughtfully. Sometimes an argument can be greatly simplified by introducing a variable, devising a special notation, or defining a new term. But do this sparingly, since you're requiring the reader to remember all this new stuff. And remember to actually define the meanings of new variables, terms, or notations; don't just start using them.

Simplify. Long, complicated proofs take the reader more time and effort to understand and can more easily conceal errors. So a proof with fewer logical steps is a better proof.

Don't bully. Words such as "clearly" and "obviously" serve no logical function. Rather, they almost always signal an attempt to bully the reader into accepting something which the author is having trouble justifying rigorously. Don't use these words in your own proofs and go on the alert whenever you read one.

Finish. At some point in a proof, you'll have established all the essential facts you need. Resist the temptation to quit and leave the reader to draw the right conclusions. Instead, tie everything together yourself and explain why the original claim follows.

2 Exercises

Exercise 2.1. *Let $\mathbb{Z}$ be the integers. For each of the following functions, $f : \mathbb{Z} \longrightarrow \mathbb{Z}$, determine whether f is (i) one-to-one, (ii) onto, and/or (iii) bijective. Prove your answer (briefly).*

1. $f(x) = x$
2. $f(x) = x^2$
3. $f(x) = \lfloor \cos(x) \rfloor$

Exercise 2.2. *Argue that "has the same birthday as" is an equivalence relation.*

Exercise 2.3. *Prove that* $\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$.

Exercise 2.4. *Use the Pigeonhole Principle to prove that at every party, there are at least two people who have shaken hands with the same number of people.*